Finite-Time Median Related Group Consensus over Directed Networks

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ABSTRACT

In this paper, we investigate a novel finite-time median related group consensus problem, where the finial consensus value can be identified as a desired function of the median of initial states instead of the much studied average value. The underlying communication topology is modeled by a weighted dynamical directed network. A distributed control protocol is firstly introduced to ensure that the agents can reach a median related consensus in finite time in a collaboration network, meaning that all edge-weights of the communication network are non-negative. We then generalize the results to cooperation-competition networks, where the communication network is divided into predetermined collaboration sub-networks allowing possibly negative weights. Effective group control protocols are designed to guarantee the median related group consensus in finite time. Finally, numerical simulations are presented to illustrate the theoretical results.

KEYWORDS
Finite-time consensus; multi-agent systems; median related consensus; collaboration networks; cooperation-competition networks.

1. Introduction

Over the last decade, consensus problems of multi-agent systems have attracted wide research interest partly due to the emergence of sensor networks, social networks, multi-agent robotics and mobile ad hoc communication. Consensus problem is one of the most fundamental problems in multi-agent coordination, where a collection of agents agree on a common state value in a distributed manner. Arguably, the biggest challenge for researchers in this area is to find a proper control protocol over the network, allowing the corresponding system to reach some form of consensus. The significance of the protocol is to provide a concise formalism for examining means by which the network topology dictates properties of the dynamic process evolving over it. More backgrounds and a variety of results on consensus problems can be found in (Mesbahi & Egerstedt, 2010).

Most of the existing work focuses on the average consensus, where the desired consensus value is the average of agents' initial states (Bliman & Trecate, 2008; Cai & Ishii, 2012; Garcia & Hadjicostis, 2013; Priolo, Gasparri, Montijiano, & Sagues, 2014; Rajagopal & Wainwright, 2011; Seyboth, Dimarogonas, & Johansson, 2013). The distributed averaging problems can be studied either in continuous-time or in discrete-time settings. In the recent study (Manfredi & Angeli, 2017), for example, researchers have exploited a suitable notion of integral connectivity to present a new result on asymptotic average consensus for continuous time non-autonomous nonlinear networks under almost-periodic interactions. However, when there are some outlier agents (i.e., the agents' initial states hold at certain abnormal level) in the system, the average consensus no longer meets our needs as the average value is sensitively affected by the outliers. One solution is to find a new suitable consensus value to replace the average. It is worth noting that the median value is a statistical term, which is particularly robust to the existence of outlier agents. Recently, (Pilloni & Pisano, 2016) has showed how the integral sliding-mode control design paradigm can be successfully applied to the framework of multi-agent systems to solve the consensus on the median value problem for a network of perturbed non-identical single integrators. The work (Franceschelli, Giua, & Pisano, 2017) studies distributed protocol that achieves consensus on the median value of the agents' initial values in finite time by exploiting a suitable ad-hoc discontinuous local interaction rule.

On the other hand, requiring an arbitrary long time to reach consensus is often unacceptable in some practical situations, such as group coordination, the cooperative tasks of the unmanned aerial vehicle and robot-soccer. Some researchers have explored the finite-time consensus (Cao & Ren, 2014; Mei, Wu, Ning, & Lu, 2016; Shang, 2017; Wang & Xiao, 2010) or fixed-time consensus (Defoort, Polyakov, Demesure, Djemai, & Veluvolu, 2015; Fu & Wang, 2016; Polyakov, 2012; Shang & Ye, 2017) accordingly. For example, in (Lin, Ren, & Farrell, 2017), the distributed optimization problem with general differentiable convex objective functions is studied, showing that all agents can reach a consensus in finite time while minimizing the team objective function asymptotically.

It is worth investigating how the corresponding problems might be solved under the cooperation-competition networks. In some practical applications, the relationship between two agents can be cooperative or competitive due to limited resources. These systems usually do not lead to complete consensus in this situation. In order to overcome the adverse effects caused by competition, there are two popular methods. One generic method is to split the agents into multiple collaboration subgraphs and to let the agents in each subgraph reach an individual consistent state in finite time. Group consensus has been studied in this context by a number of researchers(Cui, Xie, & Jiang, 2016; Han & Chen, 2015; Shang & Ye, 2017; Yu & Wang, 2012). For example, in (Shang & Ye, 2017), the distributed tracking control protocol is introduced to ensure that the follower agents in each subgraph can track their respective leaders in a prescribed time regardless of the initial conditions. Otherwise, the cooperation-competition networks are developed by benefiting from a common characteristic of their own, i.e., communications among vertices are represented by signed graphs(Zaslavsky, 1982) which admit negative edge weights in addiction to positive edge weights. The bipartite or modulus consensus of cooperative-antagonistic networks can be explored in many papers(Altafini, 2013; Meng, 2017; Meng, Shi, & Johansson, 2015; Zhang & Chen, 2017). The work (Meng, 2017) studies distributed control under hybrid static and dynamic interactions in cooperative-antagonistic networks and solves the bipartite consensus problem. There are several differences between the first and the second method. The first aims to reach group consensus to carry out different cooperative tasks and each collaboration subnetwork is predetermined. The competition and cooperation mechanism for different groups leads to the emergence of cooperation-competition networks. However, the second focuses on the properties of cooperation-competition networks, including finite-time, fixed-time analysis or dynamic distributed control design.

Motivated by the above consideration, we in this paper aim to design distributed protocols such that the multi-agent system can reach a finite-time consensus towards some median related values. We further consider the said consensus problems under collaboration networks and cooperation-competition networks, respectively. The contribution of this paper is threefold. First, a distributed consensus protocol is designed to achieve finite-time median related consensus generalizing the results in (Franceschelli et al., 2017), where the final consensus value is the exact median. In this way, we can expand the range of consensus values in a large scope compared with the existing results(Cai & Ishii, 2012; Franceschelli et al., 2017; Pilloni & Pisano, 2016). Exactly, if the desired function of median is replaced by specific equation, then any consensus value or consensus interval could be obtained. Second, the underlying communication topology is modeled as a weighted dynamical directed network under detail-balanced condition. Compared with the static undirected topology, the main difference is that the edge-weights between any two nodes are different and time-varying, which is more needed in several real world scenarios. Third, median related group consensus is achieved in finite time and explicit estimations of the settling time are obtained. The group consensus in our paper doesn't require the inter-group balanced condition, which is literally imposed on several results(Cui et al., 2016; Han & Chen, 2015; Qin & Yu, 2013), making the topology be more flexible.

The rest of the paper is organized as follows. Section 2 provides some preliminaries and formulates on graph theory, non-smooth analysis and consensus definitions. Section 3 investigates the finite-time median related consensus problem over collaboration networks. Section 4 shows that under cooperation-competition networks, every predetermined collaboration sub-network can reach a finite-time median related consensus. Some numerical examples are given in section 5 to illustrate the theoretical results in our paper. A conclusion is drawn in section 6.

2. Preliminaries

To start with, we fix some standard notations that will be used throughout the paper. The cardinality of a set S is denoted by $|S|$. $\mu(Z) = 0$ indicates that Z is a subset of $\mathbb{R}^n$ with measure zero. Let $\mathrm{co}\{X\}$ denote the convex hull of a set X. $B(x, \delta)$ is a circle of radius δ centered around x. Let ∇V denote the conventional gradient of V. The maximum and minimum elements of the vector α are denoted by $\alpha_{\max}$ and $\alpha_{\min}$, respectively. $1_n \in \mathbb{R}^n$ is a vector with all the entries being 1. We define the signum function and the set-valued SIGN function as follows:

$$\mathrm{sign}(y) = \begin{cases} 1, & \text{if } y > 0 \ , \\ 0, & \text{if } y = 0 \ , \\ -1, & \text{if } y < 0 \ , \end{cases}$$

$$\mathrm{SIGN}(y) = \begin{cases} 1, & \text{if } y > 0 \ , \\ [-1, 1], & \text{if } y = 0 \ , \\ -1, & \text{if } y < 0 \ . \end{cases}$$

Definition 2.1. A function $g(\cdot)$ is called order-keeping in an interval $I \subseteq \mathbb{R}$, if $g(\cdot)$ is monotone increasing or monotone decreasing within the interval I.

2.1. Graph Theory

The communication network of a multi-agent system can often be described by a weighted dynamic directed graph. At time t, let $G(t) = (V, E(t))$ be a weighted directed graph, where the node set $V = \{1, 2, \ldots, n\}$ represents n agents and the edge set $E(t) \subseteq V \times V$ describes the information exchange among the agents. Here, the associated weighted adjacency matrix of the graph is denoted by $W(t) = (w_{ij}(t)) \in \mathbb{R}^{n\times n}$. $E(t)$ satisfies that

$$\begin{cases} (j,i) \in E(t), & \text{if } w_{ij}(t) \neq 0 \ , \\ (j,i) \notin E(t), & \text{if } w_{ij}(t) = 0 \ . \end{cases}$$

Let $N_i(t) = \{j \in V : (j,i) \in E(t)\}$ be the set of neighbors of agent i at time t.

The term 'labeled graph' when used without qualification means a graph with each node labeled differently, so that all nodes are considered distinct for the purposes of enumeration. When we delete the 'labels' on the vertices, the graph is called unlabeled in this case. The underlying graph of a digraph is the graph obtained by replacing each ordered vertex pair of digraph with unordered vertex pair. A graph G is connected if there is a path in G between any given pair of vertices, otherwise it is disconnected. Every disconnected graph can be split up into a number of connected subgraphs, called components.

Definition 2.2. (Mesbahi & Egerstedt, 2010). An edge cut-set in G is the set of edges whose deletion increases the number of connected components of G. The edge connectivity of the graph G, denoted by ρ , is the minimum number of edges in any of its edge cut-sets.

Actually, the edge connectivity is an important notion with specific applications in graph theory(Bondy & Murty, 2008). There is another statement for the undirected graph that it's the maximum value of k for which G is k-edge-connected.

Definition 2.3. (Zheng & Wang, 2012). The directed graph $G(W)$ is said to satisfy the detail-balanced condition if there exist some scalars $a_i > 0$ such that $a_i w_{ij} = a_j w_{ji}$ for all $i, j \in V$. Therefore, if $(i,j) \in E$, then there is $(j,i) \in E$.

To explore the group consensus, a grouping $\mathcal{G}(t) = \{\mathcal{G}_1(t), \ldots, \mathcal{G}_K(t)\}$ of the graph $G(t)$ is defined by dividing its node set into disjoint subgraphs $\{\mathcal{G}_k(t)\}_{k=1}^K$ at each time instant t. In other words, $\mathcal{G}(t)$ satisfies $\cup_{k=1}^K \mathcal{G}_k(t) = V$ and $\mathcal{G}_k(t) \cap \mathcal{G}_{k'}(t) = \varnothing$ for $k \neq k'$. We write $\mathcal{G}_k(t) = (V_k, E_k(t))$ and $|V_k| = n_k$. Hence, $n = n_1 + \cdots + n_K$. We assume that the interactions between agents in the same subgraph are cooperative; namely, $w_{ij}(t) > 0$ if $(i,j) \in E_k(t)$ for every k at any time. The interactions between agents in the different subgraph can be either non-negative (i.e., cooperative) or non-positive (i.e., competitive). Each subgraph $\mathcal{G}_k(t)(1 \leq k \leq K)$ inherits the structure of $G(t)$ naturally in the sense of induced subgraph at time t. For each $1 \leq k \leq K$, the associated weighted adjacency matrix of $\mathcal{G}_k(t)$ is similarly defined as $W_k(t) = (w_{ij}(t)) \in \mathbb{R}^{n_k\times n_k}$.

2.2. Non-Smooth Analysis

Consider the (possibly discontinuous) dynamical system

$$\begin{aligned} \dot{x} &= f(x), \quad x \in \mathbb{R}^n, \\ x(0) &= x_0, \quad x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $f(x) : \mathbb{R}^n \to \mathbb{R}^n$, is defined almost everywhere, i.e., it is defined for every $x \in \mathbb{R}^n \backslash Z$, where $\mu(Z) = 0$. Furthermore, $f(x)$ is measurable in an open region $Q \subset \mathbb{R}^n$ and for all compact sets $D \subset Q$ there is a constant A_D such that $||f(x)|| \leq A_D$ almost everywhere in D.

Definition 2.4. A vector function $x(\cdot) \in \mathbb{R}^n$ is called a Filippov solution of (1) on $[t_0, t_1]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_1]$ and, for almost all $t \in [t_0, t_1]$, satisfies the differential inclusion

$$\dot{x} \in K(x) \triangleq \bigcap_{\delta>0} \bigcap_{\mu(N)=0} \mathrm{co}\left\{ f(B(x, \delta) \backslash N, t) \right\}. \tag{2}$$

Definition 2.5. Let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. Its Clarke's generalized gradient $\partial V(x)$ is defined as

$$\partial V(x) \triangleq \mathrm{co}\left\{ \lim_{i\to\infty} \nabla V(x_i) | x_i \to x, x_i \notin \Omega_V \cup N \right\},$$

where Ω_V is a set of Lebesgue measure zero which contains all points where $\nabla V(x)$ does not exist, and N is an arbitrary set which satisfies $\mu(N) = 0$.

Definition 2.6. Given a locally Lipschitz function $V(x)$, where $x \in \mathbb{R}^n$ is governed by the differential inclusion $\dot{x} \in K(x)$, the set-valued Lie derivative of $V(x)$ at x is

$$\tilde{\mathcal{L}}V(x) = \{ a \in \mathbb{R} | \exists v \in K(x) \text{ such that } \zeta \cdot v = a, \forall \zeta \in \partial V(x) \}.$$

It follows that $\tilde{\mathcal{L}}V(x) = K(x) \times \partial V(x)$. As mentioned in (Cortes, 2008), the set-valued Lie derivative allows the study of the evolution of a Lyapunov function according to the next lemma.

Lemma 2.7. *Evolution along Filippov solutions (Clarke, 1983). Let $x(t) : [t_0, t_1] \to \mathbb{R}^n$ be a Filippov solution of (2). Let $V(x)$ be a locally Lipschitz and regular function. Then $dV(x(t))/dt$ exists a.e. and $dV(x(t))/dt \in \tilde{\mathcal{L}}V(x(t))$ a.e..*

2.3. Finite-Time Median Related Consensus

We consider the following multi-agent system with n agents governed by

$$\begin{aligned} \dot{x}_i(t) &= u_i(t), \quad i = 1, 2, \ldots, n, \\ x_i(0) &= z_i, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{3}$$

where $x_i(t) \in \mathbb{R}$ is the state of agent i at time t and $u_i(t) \in \mathbb{R}$ is the control input of agent i at time t. Let $z = [z_1, z_2, \ldots, z_n]$ be the agents' initial states. The original graph is regarded as unlabeled graph and relabel the vertices such that

$$z_i \leq z_{i+1}, \quad i = 1, 2, \ldots, n-1.$$

Note that at time t, every cooperation-competition network can be regarded as a grouping $\mathcal{G}(t) = \{\mathcal{G}_1(t), \ldots, \mathcal{G}_K(t)\}$, where $\mathcal{G}_k(t)(1 \leq k \leq K)$ is a collaboration subnetwork. The agents' initial states in $\mathcal{G}_k(t)$ are denoted by $z^k = [z_1^k, z_2^k, \ldots, z_{n_k}^k]$, which satisfies

$$z_i^k \leq z_{i+1}^k, \quad i = 1, 2, \ldots, n_k - 1.$$

Definition 2.8. Let the vector z be in ascending order and $|z| = n$. Then the median value $m(z)$ takes the form

$$m(z) \in \begin{cases} \left\{ z_{\frac{n+1}{2}} \right\}, & \text{if } n \text{ is odd,} \\ \left[z_{\frac{n}{2}}, z_{\frac{n}{2}+1} \right], & \text{if } n \text{ is even.} \end{cases}$$

We aim to design a decentralized consensus protocol such that the system can reach a finite-time median related consensus. Our strategy is as follows. We first prove that the multi-agent system can reach a finite-time consensus. Then we show that the consensus function converges to a desired function of median in finite-time.

Definition 2.9. For a given function $g(\cdot)$,

(1) The multi-agent system is said to achieve finite-time median related consensus if there are $T' > T > 0$ and $c(t) \in \mathbb{R}$ such that

$$\begin{aligned} x_i(t) &= c(t), \quad \forall i \in V, \quad t \geq T, \\ c(t) &= g\big(m(z)\big), \quad \forall t \geq T', \end{aligned}$$

where $c(t)$ is referred to as 'consensus function'.

(2) The multi-agent system is said to achieve finite-time median related group consensus if every subgraph achieves the corresponding finite-time median related consensus. That is, there is $T_{max} > 0$ such that

$$x_i(t) = g\big(m(z^k)\big), \quad \forall i \in V_k, k = 1, \ldots, K, \quad t \geq T_{max}.$$

Remark 1. If n is odd, $c(t) = g(m(z)), \forall t \geq T'$ is uniquely defined; namely, the consensus function converges to a constant finally. Whereas $g(m(z))$ belongs to the closed interval $\left[g\big(z_{\frac{n}{2}}\big), g\big(z_{\frac{n}{2}+1}\big)\right]$ when n is even; namely, the median related consensus is achieved if $c(t) \in \left[g\big(z_{\frac{n}{2}}\big), g\big(z_{\frac{n}{2}+1}\big)\right], \forall t \geq T'$.

To simplify proof, we provide a generalization of an extended Lyapunov Lemma for non-smooth analysis, which has been proven in (Franceschelli et al., 2017) and (Paden & Sastry, 1987).

Lemma 2.10. *Let $M = span(1_n)$ be the subspace spanned by vector 1_n. Consider a scalar function $V(x) : \mathbb{R}^n \to \mathbb{R}$, with $V(x) = 0$ $\forall x \in M$ and $V(x) > 0$ $\forall x \notin M$. Let $x : \mathbb{R} \to \mathbb{R}^n$ and $V(x(t))$ be absolutely continuous on $[t_0, \infty)$ with $dV(x(t))/dt \leq -\epsilon < 0$*

a.e.on $\{t|x(t) \notin M\}$. Then, $V(x(t))$ converges to 0 in finite time and $x(t)$ reaches the subspace M in finite time as well.

3. Finite-Time Consensus over Collaboration Networks

In this section, motivated by the finite-time control techniques used in (Franceschelli et al., 2017), we design the following distributed controller for agents in order to reach a finite-time median related consensus:

$$u_i(t) = -\alpha_i \text{sign}(x_i(t) - g(z_i)) - \sum_{j \in N_i(t)} a_i(t) w_{ij}(t) \text{sign}(x_i(t) - x_j(t)), \tag{4}$$

where $\alpha_i \in \mathbb{R}^+, \forall i \in V$, $g(\cdot)$ is the given function of Definition 2.9, $a_i(t) \in \mathbb{R}^+, \forall i \in V$ and $w_{ij}(t)$ is the element of weighted adjacent matrix in the ith row and the jth column at time t. Note that only local information between neighboring agents is needed.

Assumption 1. We assume that

(1) In the protocol (4), $a_i(t) \geq L_1 > 0$ for all $i \in V$ and $t \geq 0$, where L_1 is a constant.
(2) In the weighted adjacency matrix $W(t)$ of $G(t)$, $w_{ij}(t) \geq L_2 > 0$ for all $(i,j) \in E(t)$ and $t \geq 0$, where L_2 is a constant.
(3) In the weighted dynamic directed graph $G(t)$, $\exists \rho \geq 1$ such that the edge connectivity $\rho(t) \geq \rho$ for all $t \geq 0$, where ρ is a constant.

Remark 2. Under a collaboration network, Assumption 1 can be constructed obviously due to the existence of their greatest lower bounds. Without any constraints, L_1, L_2 and ρ can be set to the minimum values of $a_i(t)$, $w_{ij}(t)$ and $\rho(t)$, respectively.

As we have mentioned above, here we verify the following two theorems to support that the system with given protocol (4) can reach a finite-time median related consensus over collaboration networks.

Theorem 3.1. *Consider the networked system (3) along with a weighted dynamic directed graph $G(t), t \geq 0$. Under Assumption 1, let the local interaction rule be implemented as*

$$0 < \alpha_{max} < \frac{2\rho L_1 L_2}{n},$$

then the system with protocol (4) achieves consensus in a finite time and the transient time T satisfies that

$$T \leq T_1 = \frac{\max_{i \in V} x_i(0) - \min_{i \in V} x_i(0)}{\mu^2},$$

$$\mu^2 = 2\Big(\frac{2\rho L_1 L_2}{n} - \alpha_{max}\Big).$$

Proof. Let

$$I_{\max}(t) = \left\{ r \in V : x_r = \max_{i \in V} x_i(t) \right\},$$
$$I_{\min}(t) = \left\{ r \in V : x_r = \min_{i \in V} x_i(t) \right\}.$$

Consider the non-smooth Lyapunov candidate function

$$V_1(x(t)) = \frac{\sum\limits_{i \in I_{\max}(t)} x_i(t)}{|I_{\max}(t)|} - \frac{\sum\limits_{i \in I_{\min}(t)} x_i(t)}{|I_{\min}(t)|}. \tag{5}$$

It is clear that $V_1(x(t)) \geq 0$ and $V_1(x(t)) = 0$ if and only if $\max\limits_{i \in V}(x_i) = \min\limits_{i \in V}(x_i)$; namely, the network is at consensus.

We note that the cardinalities of sets $I_{\max}(t)$ and $I_{\min}(t)$ change over time. But fortunately, these can be treated as constants while evaluating the generalized gradient of $V_1(x(t))$ according to (Franceschelli et al., 2017). Because the cardinalities of the sets $I_{\max}(t)$ and $I_{\min}(t)$ are both piecewise constant functions whose instants of discontinuity belong to a set of measure zero. Therefore, we replace $I_{\max}(t)$ and $I_{\min}(t)$ with $I_{\max}$ and $I_{\min}$, respectively. Furthermore, $V_1(x(t)) : \mathbb{R}^n \to \mathbb{R}$ is absolutely continuous because the function $V_1(\cdot)$ is locally Lipschitz continuous and $x(t)$ is absolutely continuous thanks to the composition of $\dot{x}_i(t)$. Therefore, there is $\mu(N) = 0$ such that for all $t \in [0, \varnothing)\backslash N$, both $\dot{x}_i(t)$ and the generalized time derivative $d/dt(V_1(x(t)))$ exist.

Now, fix $t \in [0, \infty)\backslash N$. Then

$$\frac{d}{dt}(V_1(x(t))) = \frac{\sum\limits_{i \in I_{\max}} \dot{x}_i(t)}{|I_{\max}|} - \frac{\sum\limits_{i \in I_{\min}} \dot{x}_i(t)}{|I_{\min}|}. \tag{6}$$

One straightforwardly derives

$$\begin{aligned}
\sum_{i \in I_{\max}} \dot{x}_i(t) = &\sum_{i \in I_{\max}} \Big(-\alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big) \\
&- \sum_{j \in N_i(t)} a_i(t) w_{ij}(t) \mathrm{sign}\big(x_i(t) - x_j(t)\big)\Big) \\
= &\sum_{i \in I_{\max}} \Big(-\alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big) \\
&- \sum_{j \in N_i(t) \cap I_{\max}} a_i(t) w_{ij}(t) \mathrm{sign}\big(x_i(t) - x_j(t)\big) \\
&- \sum_{j \in N_i(t) \backslash I_{\max}} a_i(t) w_{ij}(t) \mathrm{sign}\big(x_i(t) - x_j(t)\big)\Big).
\end{aligned}$$

We can notice that $\forall i, j \in I_{\max}$, $\mathrm{sign}(x_i(t) - x_j(t)) = 0$ and $\forall i \in I_{\max}, j \notin I_{\max}$,

$\text{sign}(x_i(t) - x_j(t)) = 1$, then

$$\sum_{i \in I_{\max}} \dot{x}_i(t) = \sum_{i \in I_{\max}} \Big(-\alpha_i \text{sign}\big(x_i(t) - g(z_i)\big) - \sum_{j \in N_i(t) \setminus I_{\max}} a_i(t) w_{ij}(t) \Big). \tag{7}$$

As the same operation applies to $\sum_{i \in I_{\min}} \dot{x}_i(t)$, we can get that

$$\sum_{i \in I_{\min}} \dot{x}_i(t) = \sum_{i \in I_{\min}} \Big(-\alpha_i \text{sign}\big(x_i(t) - g(z_i)\big) + \sum_{j \in N_i(t) \setminus I_{\min}} a_i(t) w_{ij}(t) \Big). \tag{8}$$

Substituting (7) and (8) into (6), one obtains that $\frac{d}{dt}(V_1(x(t)))$ takes values in the following set-valued map:

$$\begin{aligned}
\frac{d}{dt}(V_1(x(t))) \in & \frac{1}{|I_{\max}|} \sum_{i \in I_{\max}} \Big(-\alpha_i \text{SIGN}\big(x_i(t) - g(z_i)\big) \\
& - \sum_{j \in N_i(t) \setminus I_{\max}} a_i(t) w_{ij}(t) \Big) \\
& - \frac{1}{|I_{\min}|} \sum_{i \in I_{\min}} \Big(-\alpha_i \text{SIGN}\big(x_i(t) - g(z_i)\big) \\
& + \sum_{j \in N_i(t) \setminus I_{\min}} a_i(t) w_{ij}(t) \Big) \\
\leq & \alpha_{\max} - \frac{\rho L_1 L_2}{|I_{\max}|} + \alpha_{\max} - \frac{\rho L_1 L_2}{|I_{\min}|}.
\end{aligned} \tag{9}$$

Next, we will estimate the righthand side of (9). Let $I_{\max} = p \leq n$, then $I_{\min} \leq n - p$. The upper bound is maximized taking $p = \frac{n}{2}$, therefore it yields

$$\begin{aligned}
\frac{d}{dt}(V_1(x(t))) &\leq 2\alpha_{\max} - \frac{n\rho L_1 L_2}{p(n-p)} \\
&\leq 2\alpha_{\max} - \frac{4\rho L_1 L_2}{n}.
\end{aligned}$$

Let $\mu^2 = 2(\frac{2\rho L_1 L_2}{n} - \alpha_{\max}) > 0$. We have

$$\frac{d}{dt}(V_1(x(t))) \leq -\mu^2,$$

which shows the finite-time convergence of (6) to zero according to Lemma 2.10. Since

$$\begin{aligned} V_1(x(t)) =& V_1(x(0)) + \int_0^t \frac{d}{dt}(V_1(x(t)))dt \\ \leq & \max_{i\in V} x_i(0) - \min_{i\in V} x_i(0) - \mu^2 t, \end{aligned}$$

there is $T \leq T_1 = \frac{\max_{i\in V} x_i(0) - \min_{i\in V} x_i(0)}{\mu^2}$ such that $\forall i \in V, x_i(t) = c(t), \forall t \geq T$, where $\mu^2 = 2(\frac{2\rho L_1 L_2}{n} - \alpha_{\max})$. □

Remark 3. Recall the definition of the edge connectivity, Theorem 3.1 indicates that one sufficient condition for finite-time consensus is the connectivity of $G(t)$ for all $t \geq 0$. It also fits for the static network, where the connectivity is an important consideration being consensus(Mesbahi & Egerstedt, 2010). Moreover, we choose $\rho(t) \geq \rho \geq 1$ instead of $\rho(t) \geq 1$ in Assumption 1. The reason lies in, the convergence time upper bound T_1 is dependent with ρ and it decreases with the increase of ρ.

Theorem 3.2. *Consider the networked system (3) along with a weighted dynamic directed graph $G(t), t \geq 0$. Assume that $\forall t \geq 0, G(t)$ satisfies the detail-balanced condition and $g(\cdot)$ is order-keeping in $co\{z\}$. Then under Assumption 1, let the local interaction rule be implemented as*

$$0 < \alpha_{max} < \frac{2\rho L_1 L_2}{n},$$
$$\frac{\alpha_{max} - \alpha_{min}}{\alpha_{min}} \leq \frac{1}{n},$$

then $\exists T_2 \geq T$ such that the consensus function $c(T)$ of Theorem 3.1 converges to $g(m(z))$, where

$$T_2 \leq 2n\frac{|c(T) - g(m(z))|}{\alpha_{max}} + T.$$

Proof. Consider the Lyapunov function

$$V_2(x(t)) = |c(t) - g(m(z))|, \tag{10}$$

then the corresponding generalized gradient is

$$\partial V_2(x(t)) = \mathrm{SIGN}\Big(c(t) - g\big(m(z)\big)\Big).$$

First, we notice that $\forall i \in V, x_i(t) = c(t), \forall t \geq T$ and hence $c(t) = \frac{\sum_{i\in V} x_i(t)}{n}, \forall t \geq T$.

Then

$$
\begin{aligned}
\dot{c}(t) = \frac{\sum_{i\in V} \dot{x}_i(t)}{n} &= -\frac{\sum_{i\in V} \alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big)}{n} \\
&\quad - \frac{\sum_{i\in V} \sum_{j\in N_i(t)} a_i(t) w_{ij}(t) \mathrm{sign}\big(x_i(t) - x_j(t)\big)}{n} \\
&= -\frac{\sum_{i\in V} \alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big)}{n} \\
&\in -\frac{1}{n} \sum_{i\in V} \alpha_i \mathrm{SIGN}\big(x_i(t) - g(z_i)\big),
\end{aligned} \tag{11}
$$

where the third equation is due to the detail-balanced condition. That is,

$$
\begin{aligned}
&\sum_{i\in V} \sum_{j\in N_i(t)} a_i(t) w_{ij}(t) \mathrm{sign}\big(x_i(t) - x_j(t)\big) \\
=&\sum_{i\in V} \sum_{j\in N_i(t)} \Big(\frac{1}{2} a_i(t) w_{ij}(t) \mathrm{sign}\big(x_i(t) - x_j(t)\big) \\
&+ \frac{1}{2} a_j(t) w_{ji}(t) \mathrm{sign}\big(x_j(t) - x_i(t)\big)\Big) \\
=&\sum_{i\in V} \sum_{j\in N_i(t)} \frac{1}{2}\big(a_i(t) w_{ij}(t) - a_j(t) w_{ji}(t)\big) \mathrm{sign}\big(x_i(t) - x_j(t)\big) \\
=&0
\end{aligned} \tag{12}
$$

Let

$$
\begin{aligned}
I_{\text{up}} &= \{s \in V : x_s < g(z_s)\}, \\
I_{\text{down}} &= \{s \in V : x_s > g(z_s)\}, \\
I_{\text{equal}} &= \{s \in V : x_s = g(z_s)\},
\end{aligned}
$$

then we can rewrite (11) as follows:

$$
\begin{aligned}
\dot{c}(t) \in -\frac{1}{n}\Big(&\sum_{i\in I_{\text{up}}} \alpha_i \mathrm{SIGN}\big(x_i(t) - g(z_i)\big) \\
&+ \sum_{i\in I_{\text{down}}} \alpha_i \mathrm{SIGN}\big(x_i(t) - g(z_i)\big) \\
&+ \sum_{i\in I_{\text{equal}}} \alpha_i \mathrm{SIGN}\big(x_i(t) - g(z_i)\big)\Big).
\end{aligned}
$$

It is obvious that

$$
\begin{aligned}
\forall i \in I_{\text{down}}, \mathrm{SIGN}(x_i(t) - g(z_i)) = 1, \\
\forall i \in I_{\text{up}}, \mathrm{SIGN}(x_i(t) - g(z_i)) = -1.
\end{aligned}
$$

Thus,

$$\dot{c}(t) \in -\frac{1}{n}\left\{\sum_{i\in I_{\text{down}}}\alpha_i - \sum_{i\in I_{\text{up}}}\alpha_i + \sum_{i\in I_{\text{equal}}}\alpha_i \text{SIGN}\big(x_i(t) - g(z_i)\big)\right\}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}V_2(x(t)) = &-\frac{1}{n}\text{SIGN}\Big(c(t) - g\big(m(z)\big)\Big)\\ &\times\left\{\sum_{i\in I_{\text{down}}}\alpha_i - \sum_{i\in I_{\text{up}}}\alpha_i + \sum_{i\in I_{\text{equal}}}\alpha_i \text{SIGN}\big(x_i(t) - g(z_i)\big)\right\}.\end{aligned} \tag{13}$$

Let

$$(*) = \sum_{i\in I_{\text{down}}}\alpha_i - \sum_{i\in I_{\text{up}}}\alpha_i + \sum_{i\in I_{\text{equal}}}\alpha_i \text{SIGN}\big(x_i(t) - g(z_i)\big).$$

We note the following two points:

(1) It is always true that $\forall t \geq T$,

$$\begin{aligned}(*) &\leq \alpha_{\max}|I_{\text{down}}| - \alpha_{\min}|I_{\text{up}}| + \alpha_{\max}|I_{\text{equal}}|,\\ (*) &\geq \alpha_{\min}|I_{\text{down}}| - \alpha_{\max}|I_{\text{up}}| - \alpha_{\max}|I_{\text{equal}}|.\end{aligned} \tag{14}$$

(2)

$$\begin{aligned}c(t) &= m(g(z)) = g(m(z)), \qquad \text{if } |I_{\text{down}}| = |I_{\text{up}}|\ ,\\ c(t) &> m(g(z)) = g(m(z)), \qquad \text{if } |I_{\text{down}}| > |I_{\text{up}}|\ ,\\ c(t) &< m(g(z)) = g(m(z)), \qquad \text{if } |I_{\text{down}}| < |I_{\text{up}}|\ .\end{aligned} \tag{15}$$

Therefore, we only need to verify the situation that $|I_{\text{up}}| \neq |I_{\text{down}}|$. We consider two cases according to the relations between $|I_{\text{down}}|$, $|I_{\text{up}}|$ and $|I_{\text{equal}}|$.

Case 1. $||I_{\text{down}}| - |I_{\text{up}}|| > |I_{\text{equal}}|$.

(i) $|I_{\text{down}}| > |I_{\text{up}}|$, that is $|I_{\text{down}}| > |I_{\text{up}}| + |I_{\text{equal}}|$. In light of (14) and (15), the right hand side of (13) can be estimated as

$$\tilde{\mathcal{L}}V_2(x(t)) \leq -\frac{1}{n}(\alpha_{\min}|I_{\text{down}}| - \alpha_{\max}|I_{\text{up}}| - \alpha_{\max}|I_{\text{equal}}|).$$

Let $|I_{\text{up}}| + |I_{\text{equal}}| = p$, then $|I_{\text{down}}| = n - p$, and $n - p > p$, that is $n - p \geq p + 1$,

$p \le \frac{n-1}{2}$. Thus

$$\begin{aligned}\tilde{\mathcal{L}}V_2(x(t)) &\le -\frac{1}{n}\big(\alpha_{\min}(n-p)-\alpha_{\max}p\big)\\ &\le -\frac{1}{n}\Big(\alpha_{\min}(\frac{n+1}{2})-\alpha_{\max}\frac{n-1}{2}\Big)\\ &= -\frac{1}{n}\alpha_{\max}\Big(\frac{\alpha_{\min}}{\alpha_{\max}}\frac{n+1}{2}-\frac{n-1}{2}\Big)\\ &\le -\frac{\alpha_{\max}}{2n},\end{aligned} \tag{16}$$

where the forth inequality is because $\frac{\alpha_{\max}-\alpha_{\min}}{\alpha_{\min}} \le \frac{1}{n}$.

(ii) $|I_{\text{down}}| < |I_{\text{up}}|$, that is $|I_{\text{up}}| > |I_{\text{down}}| + |I_{\text{equal}}|$. In light of (14) and (15), the right hand side of (13) can be estimated as

$$\tilde{\mathcal{L}}V_2(x(t)) \le \frac{1}{n}(\alpha_{\max}|I_{\text{down}}| - \alpha_{\min}|I_{\text{up}}| + \alpha_{\max}|I_{\text{equal}}|).$$

Let $|I_{\text{down}}| + |I_{\text{equal}}| = p$, then $|I_{\text{up}}| = n-p$, and $n-p > p$, that is $n-p \ge p+1$, $p \le \frac{n-1}{2}$. Thus

$$\begin{aligned}\tilde{\mathcal{L}}V_2(x(t)) &\le \frac{1}{n}\big(\alpha_{\max}p-\alpha_{\min}(n-p)\big)\\ &= -\frac{1}{n}\big(\alpha_{\min}(n-p)-\alpha_{\max}p\big)\\ &\le -\frac{\alpha_{\max}}{2n}.\end{aligned} \tag{17}$$

Results from what have been discussed above suggest that

$$\tilde{\mathcal{L}}V_2(x(t)) \le -\frac{\alpha_{\max}}{2n} \tag{18}$$

holds under both of cases. Combining (18) with Lemma 2.7, we can get that

$$\frac{d}{dt}(V_2(x(t))) \le -\frac{\alpha_{\max}}{2n},$$

thus it can prove the finite-time convergence of (10) to zero according to Lemma 2.10. And

$$\begin{aligned}V_2(x(t)) &= V_2(x(T)) + \int_T^t \frac{d}{dt}(V_2(x(t)))dt\\ &\le |c(T)-g(m(z))| - \frac{\alpha_{\max}}{2n}(t-T),\end{aligned}$$

so there is $T \le T_2 \le 2n\frac{|c(T)-g(m(z))|}{\alpha_{\max}} + T$ such that $\forall t \ge T_2, c(t) = g(m(z))$.

Case 2. $||I_{\text{down}}| - |I_{\text{up}}|| \le |I_{\text{equal}}|$.

Let $S_{\text{up}} = \min\limits_{s \in I_{\text{up}}} s$, $S_{\text{down}} = \max\limits_{s \in I_{\text{down}}} s$, which satisfy that

$$S_{\text{up}} > S_{\text{down}}, \quad |I_{\text{up}}| = n - S_{\text{up}} + 1, \quad |I_{\text{down}}| = S_{\text{down}},$$
$$|I_{\text{equal}}| = n - |I_{\text{up}}| - |I_{\text{down}}| = S_{\text{up}} - S_{\text{down}} - 1.$$

(i) $|I_{\text{down}}| > |I_{\text{up}}|$, that is $|I_{\text{down}}| \leq |I_{\text{equal}}| + |I_{\text{up}}|$, then

$$S_{\text{down}} \leq n - S_{\text{down}}, \quad |I_{\text{down}}| = S_{\text{down}} \leq \frac{n}{2}.$$

One can obtain that

$$|I_{\text{up}}| < |I_{\text{down}}| \leq \frac{n}{2}, \quad S_{\text{up}} = n - |I_{\text{up}}| + 1 > \frac{n}{2} + 1.$$

Thus

$$\max_{s \in I_{\text{equal}}} s = S_{\text{up}} - 1 > \frac{n}{2},$$
$$\min_{s \in I_{\text{equal}}} s = S_{\text{down}} + 1 \leq \frac{n}{2} + 1,$$

which imply that $c(t) = m(g(z)) = g(m(z))$.

(ii) $|I_{\text{down}}| < |I_{\text{up}}|$, that is $|I_{\text{up}}| \leq |I_{\text{equal}}| + |I_{\text{down}}|$, then

$$n - S_{\text{up}} + 1 \leq S_{\text{up}} - 1, \quad S_{\text{up}} \geq \frac{n}{2} + 1.$$

One can obtain that

$$|I_{\text{up}}| = n - S_{\text{up}} + 1 \leq \frac{n}{2}, \quad S_{\text{down}} = |I_{\text{down}}| < |I_{\text{up}}| \leq \frac{n}{2}.$$

Thus

$$\max_{s \in I_{\text{equal}}} s = S_{\text{up}} - 1 \geq \frac{n}{2},$$
$$\min_{s \in I_{\text{equal}}} s = S_{\text{down}} + 1 < \frac{n}{2} + 1,$$

which also imply that $c(t) = m(g(z)) = g(m(z))$.

□

Remark 4. Note that the proof contains an equation $m(g(z)) = g(m(z))$. This is because $g(\cdot)$ is order-keeping in $\text{co}\{z\}$. As mentioned above, z satisfies the ascending order, and if $g(\cdot)$ is order-keeping in $\text{co}\{z\}$ then the position of the median value of z is equal to the position of the median value of $g(z)$. Therefore, $g(m(z)) = m(g(z))$ holds.

Remark 5. Combining Theorems 3.1 and 3.2, the weighted dynamic directed graph $G(t), t \geq 0$ converges to $g(m(z))$ under certain conditions. At this time, if we construct a suitable equation with $m(z)$, the network can converge to any desired value or desired interval. For example, let $z = [1, 2, 3, 4, 5]^T$ then $m(z) = 3$. If the desired consensus value is 5, we can arrange $g(\cdot)$ as $g(m(z)) = m(z) + 2$ or $g(m(z)) = (m(z))^2 - 4$.

4. Finite-Time Group Consensus over Cooperation-Competition Networks

In this section, we discuss the situation where the agents in a cooperation-competition network are divided into several collaboration sub-networks and the states of agents in each sub-network can reach an individual median related consensus in finite time under our proposed strategy. To achieve group consensus, we assume that each collaboration subnetwork is predetermined; namely, the vertex set of every multiple subgraph never change over time. However, every subgraph is still a weighted dynamic directed graph.

Assumption 2. We assume that

(1) In the protocol (4), $L_2 \geq a_i(t) \geq L_1 > 0$ for all $i \in V$ and $t \geq 0$, where L_1 and L_2 are both constants.
(2) In the weighted adjacency matrix $W(t)$ of $\mathcal{G}(t) = \{\mathcal{G}_k(t)\}_{k=1}^K$, $w_{ij}(t) \geq L_3 > 0$ for all $(i,j) \in \cup_{k=1}^K E_k(t)$ and $t \geq 0$. $|w_{ij}(t)| \leq L_4$ for all $(i,j) \in E(t) \backslash \cup_{k=1}^K E_k(t)$ and $t \geq 0$, where L_3 and L_4 are both constants.
(3) For every subgraph $\mathcal{G}_k(t)$, $\exists \rho_k \geq 1$ such that the edge connectivity $\rho_k(t) \geq \rho_k$ for all $t \geq 0$, where $\rho_k, k \in [1, K]$ are all constants.

Theorem 4.1. *Consider the networked system (3) along with a weighted dynamic directed graph $\mathcal{G}(t) = \{\mathcal{G}_k(t)\}_{k=1}^K, t \geq 0$. Under Assumption 2, let the local interaction rule be implemented as*

$$\rho_k L_1 L_3 > L_2 L_4 \sum_{k' \neq k} n_k n_{k'}, \quad \forall k, k' \in [1, K],$$
$$0 < \alpha_{max} < \frac{2(\rho_k L_1 L_3 - \sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4)}{n_k}, \quad \forall k, k' \in [1, K],$$

then every $\mathcal{G}_k(t)$, $t \geq 0$ with protocol (4) achieves finite-time consensus and the transient time T^k satisfies that

$$T^k \leq T_3^k = \frac{\max\limits_{i \in V_k} x_i(0) - \min\limits_{i \in V_k} x_i(0)}{\beta^2},$$
$$\beta^2 = 2\left(\frac{2(\rho_k L_1 L_3 - \sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4)}{n_k} - \alpha_{max}\right).$$

Proof. Let

$$I_{\max}^k(t) = \left\{ r \in V_k : x_r = \max_{i \in V_k} x_i(t) \right\},$$
$$I_{\min}^k(t) = \left\{ r \in V_k : x_r = \min_{i \in V_k} x_i(t) \right\}.$$

Consider the non-smooth Lyapunov function

$$V_3^k(x(t)) = \frac{\sum\limits_{i\in I_{\max}^k(t)} x_i(t)}{|I_{\max}^k(t)|} - \frac{\sum\limits_{i\in I_{\min}^k(t)} x_i(t)}{|I_{\min}^k(t)|}. \tag{19}$$

Here, $I_{\max}^k(t)$ and $I_{\min}^k(t)$ can be regarded as constants due to the analysis in Theorem 3.1. Then

$$\frac{d}{dt}V_3^k(x(t)) = \frac{\sum\limits_{i\in I_{\max}^k} \dot{x}_i(t)}{|I_{\max}^k|} - \frac{\sum\limits_{i\in I_{\min}^k} \dot{x}_i(t)}{|I_{\min}^k|}, \tag{20}$$

where

$$\begin{aligned}
\sum_{i\in I_{\max}^k} \dot{x}_i(t) &= \sum_{i\in I_{\max}^k} \Big(-\alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big) \\
&\quad - \sum_{j\in N_i(t)} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t) - x_j(t)\big)\Big) \\
&= \sum_{i\in I_{\max}^k} \Big(-\alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big) \\
&\quad - \sum_{j\in N_i(t)\cap I_{\max}^k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t) - x_j(t)\big) \\
&\quad - \sum_{j\in N_i(t)\cap V_k\backslash I_{\max}^k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t) - x_j(t)\big) \\
&\quad - \sum_{j\in N_i(t)\backslash V_k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t) - x_j(t)\big)\Big).
\end{aligned}$$

We can notice that $\forall i,j\in I_{\max}^k$, $\mathrm{sign}(x_i(t)-x_j(t))=0$ and $\forall i\in I_{\max}^k, j\in N_i\cap V_k\backslash I_{\max}^k$, $\mathrm{sign}(x_i(t)-x_j(t))=1$, then

$$\begin{aligned}
\sum_{i\in I_{\max}^k} \dot{x}_i(t) = \sum_{i\in I_{\max}^k} \Big(&-\alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big) - \sum_{j\in N_i(t)\cap V_k\backslash I_{\max}^k} a_i(t)w_{ij}(t) \\
&- \sum_{j\in N_i(t)\backslash V_k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t) - x_j(t)\big)\Big).
\end{aligned} \tag{21}$$

As the same operation applies to $\sum\limits_{i\in I_{\min}^k} \dot{x}_i(t)$, we can obtain

$$\begin{aligned}
\sum_{i\in I_{\min}^k} \dot{x}_i(t) = \sum_{i\in I_{\min}^k} \Big(&-\alpha_i \mathrm{sign}\big(x_i(t) - g(z_i)\big) + \sum_{j\in N_i(t)\cap V_k\backslash I_{\min}^k} a_i(t)w_{ij}(t) \\
&- \sum_{j\in N_i(t)\backslash V_k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t) - x_j(t)\big)\Big).
\end{aligned} \tag{22}$$

Substituting (21) and (22) into (20), one obtains that $\frac{d}{dt}(V_3^k(x(t)))$ takes values in the following set-valued map:

$$
\begin{aligned}
\frac{d}{dt}(V_3^k(x(t))) \in & \frac{1}{|I_{\max}^k|} \sum_{i\in I_{\max}^k} \Big(-\alpha_i \mathrm{SIGN}\big(x_i(t)-g(z_i)\big) \\
& - \sum_{j\in N_i(t)\cap V_k\setminus I_{\max}^k} a_i(t)w_{ij}(t) \\
& - \sum_{j\in N_i(t)\setminus V_k} a_i(t)w_{ij}(t)\mathrm{SIGN}\big(x_i(t)-x_j(t)\big)\Big) \\
& - \frac{1}{|I_{\min}^k|} \sum_{i\in I_{\min}^k} \Big(-\alpha_i \mathrm{SIGN}\big(x_i(t)-g(z_i)\big) \\
& + \sum_{j\in N_i(t)\cap V_k\setminus I_{\min}^k} a_i(t)w_{ij}(t) \\
& - \sum_{j\in N_i(t)\setminus V_k} a_i(t)w_{ij}(t)\mathrm{SIGN}\big(x_i(t)-x_j(t)\big)\Big) \\
\le & \alpha_{\max} - \frac{\rho_k L_1 L_3 - \sum_{k'\neq k} n_k n_{k'} L_2 L_4}{|I_{\max}^k|} \\
& + \alpha_{\max} - \frac{\rho_k L_1 L_3 - \sum_{k'\neq k} n_k n_{k'} L_2 L_4}{|I_{\min}^k|}.
\end{aligned}
\tag{23}
$$

Next, we will estimate the righthand side of (23). Let $I_{\max}^k = p \le n_k$, and then $I_{\min}^k \le n_k - p$. The upper bound is maximized taking $p = \frac{n_k}{2}$ and we have

$$
\begin{aligned}
\frac{d}{dt}(V_3^k(x(t))) &\le 2\alpha_{\max} - \frac{n_k\big(\rho_k L_1 L_3 - \sum_{k'\neq k} n_k n_{k'} L_2 L_4\big)}{p(n_k-p)} \\
&\le 2\alpha_{\max} - \frac{4\big(\rho_k L_1 L_3 - \sum_{k'\neq k} n_k n_{k'} L_2 L_4\big)}{n_k}.
\end{aligned}
$$

Let $\beta^2 = 2\left(\frac{2\left(\rho_k L_1 L_3 - \sum_{k'\neq k} n_k n_{k'} L_2 L_4\right)}{n_k} - \alpha_{\max}\right) > 0$. Hence,

$$
\frac{d}{dt}(V_3^k(x(t))) \le -\beta^2,
$$

and the finite-time convergence of (19) to zero follows according to Lemma 2.10. Since

$$
\begin{aligned}
V_3^k(x(t)) =& V_3^k(x(0)) + \int_0^t \frac{d}{dt}(V_3^k(x(t)))dt \\
\le& \max_{i\in V_k} x_i(0) - \min_{i\in V_k} x_i(0) - \beta^2 t,
\end{aligned}
$$

for every $\mathcal{G}_k(t)$, $t\ge 0$, there is $T^k \le T_3^k = \frac{\max_{i\in V_k} x_i(0) - \min_{i\in V_k} x_i(0)}{\beta^2}$ such that $\forall i \in V_k, x_i(t) =$

$c^k(t), \forall t \geq T^k$, where $\beta^2 = 2\left(\frac{2\left(\rho_k L_1 L_3 - \sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4\right)}{n_k} - \alpha_{\max}\right)$. $\square$

Remark 6. Recall that the interaction between different groups is allowed to be cooperation or competitive. Theorem 4.1 indicates that one sufficient condition for finite-time group consensus is the boundedness in terms of the edge-weights between different subgroups. This condition is less restrictive than most of the literature concerning group consensus; see, e.g., (Cui et al., 2016; Han & Chen, 2015; Qin & Yu, 2013), which requiring the inter-group balance condition; namely, $\sum\limits_{j \in \mathcal{G}_{k'}} a_{ij} = 0$ for all $i \in \mathcal{G}_k$ and $k \neq k'$.

Theorem 4.2. *Consider the networked system (3) along with a weighted dynamic directed graph $\mathcal{G}(t) = \{\mathcal{G}_k(t)\}_{k=1}^K, t \geq 0$. Assume that $\forall t \geq 0$, $\mathcal{G}_k(t)$ satisfies the detail-balanced condition and $g(\cdot)$ is order-keeping in $co\{z_k\}, \forall k \in [1, K]$. Then under Assumption 2, let the local interaction rule be implemented as*

$$\rho_k L_1 L_3 > (1 + n_k) L_2 L_4 \sum_{k' \neq k} n_k n_{k'}, \quad \forall k, k' \in [1, K],$$

$$2L_2 L_4 \sum_{k' \neq k} n_k n_{k'} < \alpha_{max} < \frac{2(\rho_k L_1 L_3 - \sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4)}{n_k}, \quad \forall k, k' \in [1, K],$$

$$\frac{\alpha_{max} - \alpha_{min}}{\alpha_{min}} \leq \frac{1}{n_k}, \quad \forall k \in [1, K],$$

then for every $\mathcal{G}_k(t)$, $t \geq 0$, $\exists T_4^k \geq T^k$ such that the consensus function $c^k(t)$ of Theorem 4.1 converges to $g(m(z^k))$, where

$$T_4^k \leq \frac{|c^k(T^k) - g(m(z^k))|}{\xi^2} + T^k,$$

$$\xi^2 = \frac{\alpha_{max} - 2L_2 L_4 \sum\limits_{k' \neq k} n_k n_{k'}}{2n_k}.$$

Proof. Consider the Lyapunov function

$$V_4^k(x(t)) = |c^k(t) - g(m(z^k))|, \tag{24}$$

then the corresponding generalized gradient is

$$\partial V_4^k(x(t)) = \mathrm{SIGN}\Big(c^k(t) - g\big(m(z^k)\big)\Big).$$

First, we notice that $\forall i \in V_k, x_i(t) = c^k(t), \forall t \geq T^k$, and that $c^k(t) = \frac{\sum_{i \in V_k} x_i(t)}{n_k}, \forall t \geq T^k$.

Then

$$
\begin{aligned}
\dot{c}^k(t) =& \frac{\sum\limits_{i\in V_k} \dot{x}_i(t)}{n_k} \\
=& -\frac{\sum\limits_{i\in V_k} \alpha_i \mathrm{sign}\big(x_i(t)-g(z_i)\big)}{n_k} \\
& -\frac{\sum\limits_{i\in V_k}\sum\limits_{j\in N_i(t)} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t)-x_j(t)\big)}{n_k} \\
=& -\frac{\sum\limits_{i\in V_k} \alpha_i \mathrm{sign}\big(x_i(t)-g(z_i)\big)}{n_k} \\
& -\frac{\sum\limits_{i\in V_k}\sum\limits_{j\in N_i(t)\cap V_k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t)-x_j(t)\big)}{n_k} \\
& -\frac{\sum\limits_{i\in V_k}\sum\limits_{j\in N_i(t)\setminus V_k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t)-x_j(t)\big)}{n_k} \\
=& -\frac{\sum\limits_{i\in V_k} \alpha_i \mathrm{sign}\big(x_i(t)-g(z_i)\big)}{n_k} \\
& -\frac{\sum\limits_{i\in V_k}\sum\limits_{j\in N_i(t)\setminus V_k} a_i(t)w_{ij}(t)\mathrm{sign}\big(x_i(t)-x_j(t)\big)}{n_k},
\end{aligned}
$$

where the forth equation is due to the equation (12).
It follows that

$$
\begin{aligned}
\dot{c}^k(t) \in& -\frac{1}{n_k}\Big(\sum_{i\in V_k} \alpha_i \mathrm{SIGN}\big(x_i(t)-g(z_i)\big) \\
& + \sum_{i\in V_k}\sum_{j\in N_i(t)\setminus V_k} a_i(t)w_{ij}(t)\mathrm{SIGN}\big(x_i(t)-x_j(t)\big)\Big).
\end{aligned}
\tag{25}
$$

Let

$$
\begin{aligned}
I^k_{\mathrm{up}} &= \{s\in V_k : x_s < g(z_s)\}, \\
I^k_{\mathrm{down}} &= \{s\in V_k : x_s > g(z_s)\}, \\
I^k_{\mathrm{equal}} &= \{s\in V_k : x_s = g(z_s)\},
\end{aligned}
$$

then we rewrite (25) as follows

$$\begin{aligned}\dot{c}(t) \in -\frac{1}{n_k}\Big(&\sum_{i\in I_{\text{up}}^k}\alpha_i\text{SIGN}\big(x_i(t)-g(z_i)\big)\\&+\sum_{i\in I_{\text{down}}^k}\alpha_i\text{SIGN}\big(x_i(t)-g(z_i)\big)\\&+\sum_{i\in I_{\text{equal}}^k}\alpha_i\text{SIGN}\big(x_i(t)-g(z_i)\big)\\&+\sum_{i\in V_k}\sum_{j\in N_i(t)\setminus V_k}a_i(t)w_{ij}(t)\text{SIGN}\big(x_i(t)-x_j(t)\big)\Big).\end{aligned}$$

Obviously,

$$\begin{aligned}&\forall i\in I_{\text{down}}^k, \text{SIGN}\big(x_i(t)-g(z_i)\big)=1,\\&\forall i\in I_{\text{up}}^k, \text{SIGN}\big(x_i(t)-g(z_i)\big)=-1.\end{aligned}$$

Thus,

$$\begin{aligned}\dot{c}(t)\in -\frac{1}{n_k}\Big(&\sum_{i\in I_{\text{down}}^k}\alpha_i-\sum_{i\in I_{\text{up}}^k}\alpha_i+\sum_{i\in I_{\text{equal}}^k}\alpha_i\text{SIGN}\big(x_i(t)-g(z_i)\big)\\&+\sum_{i\in V_k}\sum_{j\in N_i(t)\setminus V_k}a_i(t)w_{ij}(t)\text{SIGN}\big(x_i(t)-x_j(t)\big)\Big)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{L}}V_4^k(x(t))=&-\frac{1}{n_k}\text{SIGN}\Big(c^k(t)-g\big(m(z^k)\big)\Big)\\&\times\Big(\sum_{i\in I_{\text{down}}^k}\alpha_i-\sum_{i\in I_{\text{up}}^k}\alpha_i+\sum_{i\in I_{\text{equal}}^k}\alpha_i\text{SIGN}\big(x_i(t)-g(z_i)\big)\\&+\sum_{i\in V_k}\sum_{j\in N_i(t)\setminus V_k}a_i(t)w_{ij}(t)\text{SIGN}\big(x_i(t)-x_j(t)\big)\Big).\end{aligned}\tag{26}$$

Let

$$\begin{aligned}(\sharp)=&\sum_{i\in I_{\text{down}}^k}\alpha_i-\sum_{i\in I_{\text{up}}^k}\alpha_i+\sum_{i\in I_{\text{equal}}^k}\alpha_i\text{SIGN}\big(x_i(t)-g(z_i)\big)\\&+\sum_{i\in V_k}\sum_{j\in N_i(t)\setminus V_k}a_i(t)w_{ij}(t)\text{SIGN}\big(x_i(t)-x_j(t)\big).\end{aligned}$$

Note that

(1) It is always true that $\forall t \geq T^k$,

$$\begin{aligned}(\sharp) &\leq \alpha_{\max}|I^k_{\text{down}}| - \alpha_{\min}|I^k_{\text{up}}| + \alpha_{\max}|I^k_{\text{equal}}| + \sum_{k' \neq k} n_k n_{k'} L_2 L_4 \\ (\sharp) &\geq \alpha_{\min}|I^k_{\text{down}}| - \alpha_{\max}|I^k_{\text{up}}| - \alpha_{\max}|I^k_{\text{equal}}| - \sum_{k' \neq k} n_k n_{k'} L_2 L_4.\end{aligned} \tag{27}$$

(2) According to the proof of Theorem 3.2, when $|I^k_{\text{down}}| = |I^k_{\text{up}}|$ or $||I^k_{\text{down}}| - |I^k_{\text{up}}|| \leq |I^k_{\text{equal}}|$, $c^k(t) = m(g(z^k)) = g(m(z^k))$ holds. Therefore, next we only need to consider the situation that $||I^k_{\text{down}}| - |I^k_{\text{up}}|| > |I^k_{\text{equal}}|$.

Two cases can be considered similarly.

(i) $|I^k_{\text{down}}| > |I^k_{\text{up}}|$, that is $|I^k_{\text{down}}| > |I^k_{\text{up}}| + |I^k_{\text{equal}}|$. In light of (15) and (27), the right hand side of (26) can be estimated as

$$\begin{aligned}\tilde{\mathcal{L}}V^k_4(x(t)) &\leq -\frac{1}{n_k}\big(\alpha_{\min}|I^k_{\text{down}}| - \alpha_{\max}|I^k_{\text{up}}| - \alpha_{\max}|I^k_{\text{equal}}| - \sum_{k' \neq k} n_k n_{k'} L_2 L_4\big) \\ &\leq -\frac{\alpha_{\max}}{2n_k} + \frac{\sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4}{n_k},\end{aligned}$$

where the second inequation is due to (16).

(ii) $|I^k_{\text{down}}| < |I^k_{\text{up}}|$, that is $|I^k_{\text{up}}| > |I^k_{\text{equal}}| + |I^k_{\text{down}}|$. In light of (15) and (27), the right hand side of (26) can be estimated as

$$\begin{aligned}\tilde{\mathcal{L}}V^k_4(x(t)) &\leq \frac{1}{n_k}\big(\alpha_{\max}|I^k_{\text{down}}| - \alpha_{\min}|I^k_{\text{up}}| + \alpha_{\max}|I^k_{\text{equal}}| + \sum_{k' \neq k} n_k n_{k'} L_2 L_4\big) \\ &\leq -\frac{\alpha_{\max}}{2n_k} + \frac{\sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4}{n_k},\end{aligned}$$

where the second inequation is due to (17).

The above comments show that

$$\tilde{\mathcal{L}}V^k_4(x(t)) \leq -\frac{\alpha_{\max}}{2n_k} + \frac{\sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4}{n_k} \tag{28}$$

holds in both cases. Combining (28) with Lemma 2.7, we get

$$\frac{d}{dt}(V^k_4(x(t))) \leq -\frac{\alpha_{\max}}{2n_k} + \frac{\sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4}{n_k}.$$

Let $\xi^2 = \frac{\alpha_{\max} - 2\sum\limits_{k' \neq k} n_k n_{k'} L_2 L_4}{2n_k} > 0$, then $\frac{d}{dt}(V^k_4(x(t))) \leq -\xi^2$.

By Lemma 2.10, the finite-time convergence of (24) to zero follows. Since

$$\begin{aligned} V_4^k(x(t)) =& V_4^k(x(T^k)) + \int_{T^k}^{t} \frac{d}{dt}(V_4^k(x(t)))dt \\ &\leq |c^k(t) - g(m(z^k))| - \xi^2(t - T^k), \end{aligned}$$

for every $\mathcal{G}(t)$, $t \geq 0$, there is $T^k \leq T_4^k \leq \frac{|c^k(T^k)-g(m(z^k))|}{\xi^2} + T^k$ such that $\forall t \geq T_4^k, c^k(t) = g(m(z^k))$, where $\xi^2 = \frac{\alpha_{\max}-2\sum\limits_{k'\neq k} n_k n_{k'} L_2 L_4}{2n_k}$. □

The results of Theorems 4.1 and 4.2 can be straightforwardly generalized to Theorem 4.3.

Theorem 4.3. *Consider the networked system (3) along with a weighted dynamic directed graph $\mathcal{G}(t) = \{\mathcal{G}_k(t)\}_{k=1}^K, t \geq 0$. Assume that $\forall t \geq 0$, $\mathcal{G}_k(t)$ satisfies the detail-balanced condition and $g(\cdot)$ is order-keeping in $co\{z_k\}, \forall k \in [1, K]$. Then under Assumption 2, let the local interaction rule be implemented as*

$$\rho_k L_1 L_3 > (1 + n_k) L_2 L_4 \sum_{k'\neq k} n_k n_{k'}, \quad \forall k \in [1, K],$$

$$2L_2 L_4 \sum_{k'\neq k} n_k n_{k'} < \alpha_{max} < \frac{2(\rho_k L_1 L_3 - \sum\limits_{k'\neq k} n_k n_{k'} L_2 L_4)}{n_k}, \quad \forall k, k' \in [1, K],$$

$$\frac{\alpha_{max} - \alpha_{min}}{\alpha_{min}} \leq \frac{1}{n_k}, \quad \forall k \in [1, K],$$

then $\mathcal{G}(t) = \{\mathcal{G}_k(t)\}_{k=1}^K, t \geq 0$ with protocol (4) achieves finite-time median related group consensus and the transient time $T_{max} = \max\limits_{k\in[1,K]} T_4^k$, where

$$\begin{aligned} T_4^k &\leq \frac{|c^k(T^k) - g(m(z^k))|}{\xi^2} + T^k, \\ \xi^2 &= \frac{\alpha_{max} - 2\sum\limits_{k'\neq k} n_k n_{k'} L_2 L_4}{2n_k}, \\ T^k &\leq \frac{\max\limits_{i\in V_k} x_i(0) - \min\limits_{i\in V_k} x_i(0)}{\beta^2}, \\ \beta^2 &= 2\left(\frac{2\big(\rho_k L_1 L_3 - \sum\limits_{k'\neq k} n_k n_{k'} L_2 L_4\big)}{n_k} - \alpha_{max}\right). \end{aligned}$$

Remark 7. Notice that if $\forall (i,j) \in E(t)\backslash \cup_{k=1}^K E_k(t), w_{ij}(t) > 0$, then K sub-networks can be viewed as a total collaboration network.

(1) We only need to replace $|w_{ij}(t)| \leq L_4$ by $0 < w_{ij}(t) \leq L_4$ for all $(i,j) \in E(t)\backslash \cup_{k=1}^K E_k(t)$ and $t \geq 0$ in the Assumption 2. Then the group consensus in cooperative networks follows exactly from Theorems 4.1 and 4.2.
(2) Under the operation (1), it is essential to ensure that the constraints of Theorems 3.2 and 4.3 have no intersection in this situation. We suppose that over a total

cooperative network, every predetermined collaboration sub-network can reach an individual finite-time median related consensus. By Theorem 4.3, we obtain

$$w_{ij}(t) \geq L_3 > 0, \quad \forall (i,j) \in \cup_{k=1}^{K} E_k(t), \quad \forall t \geq 0, \tag{29}$$

$$0 < w_{ij}(t) \leq L_4, \quad \forall (i,j) \in E(t) \backslash \cup_{k=1}^{K} E_k(t), \quad \forall t \geq 0, \tag{30}$$

$$\rho_k L_1 L_3 > (1+n_k) L_2 L_4 \sum_{k' \neq k} n_k n_{k'}, \quad \forall k, k' \in [1, K], \tag{31}$$

$$\alpha_{\max} > 2 L_2 L_4 \sum_{k' \neq k} n_k n_{k'}, \quad \forall k, k' \in [1, K]. \tag{32}$$

In the light of (31),

$$L_4 < \frac{\rho_k L_1 L3}{L_2 (1+n_k) \sum\limits_{k' \neq k} n_k n_{k'}} \leq \frac{(n_k - 1) L_1 L_3}{L_2 (1+n_k) \sum\limits_{k' \neq k} n_k n_{k'}} < \frac{L_1 L_3}{L_2} < L_3. \tag{33}$$

It follows from (30) and (33) that

$$0 < w_{ij}(t) \leq L_4 < L_3, \quad \forall (i,j) \in E(t) \backslash \cup_{k=1}^{K} E_k(t), \quad \forall t \geq 0. \tag{34}$$

Thus combining (29) with (34), $\exists 0 < L' \leq L_4$ such that

$$w_{ij}(t) \geq L' > 0, \quad \forall (i,j) \in E(t). \tag{35}$$

If the whole graph can reach a finite-time median related consensus, then $\alpha_{\max}$ is upper-bounded by

$$\alpha_{\max} < \frac{2\rho L_1 L'}{n} \leq \frac{2\rho L_1 L_4}{n}$$

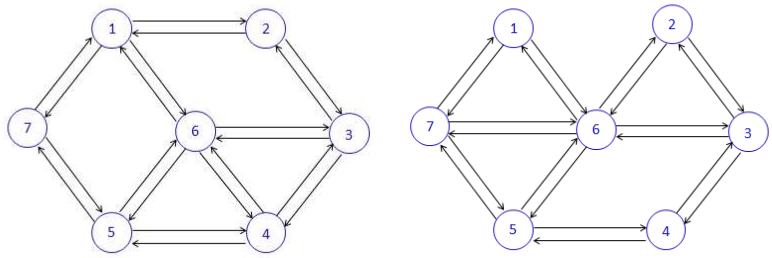
according to Theorem 3.2 and (35). It conflicts the condition $\alpha_{\max} > 2 L_2 L_4 \sum\limits_{k' \neq k} n_k n_{k'} > 2\rho L_2 L_4 > 2\rho L_1 L_4$ derived from (32). Therefore, it is clear that the constraints of Theorems 3.2 and 4.3 have no intersection over collaboration networks.

5. Numerical Examples

In this section, we present some numerical examples to illustrate our theoretical results.

Example 5.1. (Finite-time consensus over collaboration networks). In this example, we consider multi-agent system (3) with $n = 7$ agents interacting over the communication graphs in Fig. 1. To satisfy the constraint $\rho(t) \geq \rho \geq 1, \forall t \geq 0$, the system is modeled as a periodic switching graph, where the underlying communication topology is given by

$$G(t) = \begin{cases} G_1(t) = (V, E_1(t)), & \text{if } t \in [2k, 2k+1], k \in N, \\ G_2(t) = (V, E_2(t)), & \text{if } t \in [2k-1, 2k], k \in N. \end{cases}$$



(a) $G_1(t)$ (b) $G_2(t)$

Figure 1. Communication topology for Example 1. The system is modeled as a periodic switching graph, where it can switch back and forth between $G_1(t)$ and $G_2(t)$ continuously. The information exchange among nodes are both chosen as $w_{ij}(t) = i(2 + \sin(t)), \forall (i, j) \in E_1(t), E_2(t)$.

Here, the lower bound of the edge connectivities of the underlying graphs of $G_1(t)$ and $G_2(t)$ can be set to $\rho = 2$. The associated weighted adjacency matrices of $G_1(t)$ and $G_2(t)$ are both chosen as $w_{ij}(t) = i(2 + \sin(t)), \forall (i, j) \in E_1(t), E_2(t)$. The constraint $a_i(t)w_{ij}(t) = a_j(t)w_{ji}(t), \forall i \in V, \forall (i, j) \in E(t), \forall t \geq 0$ holds with $a_i(t) = \frac{7}{i(2+\sin(t))}, \forall i \in V, \forall t \geq 0$. For convenience, we regard these values $a_i(t)w_{ij}(t)$ and $a_j(t)w_{ji}(t)$ as a constant. By direct calculations, we choose $L_1 = \frac{1}{3}, L_2 = 1, \alpha_{\max} = 0.18, \alpha_{\min} = 0.16$ and $\alpha = [0.160, 0.180, 0.165, 0.170, 0.175, 0.172, 0.164]^T$ which meet all constraints of Theorem 3.2. We consider the case in which one agent holds an outlier initial value. The initial network state is chosen with values in the range $[0, 1]$, and the initial value of the outlier agent is equal to 10. The initial network state is thus

$$z = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 10]^T.$$

The initial state's average is 1.73 whereas the median value is 0.4. Here, $g(\cdot)$ is chosen as $g(m(z)) = (m(z))^2$; it is order-keeping in $\text{co}\{z\}$. Combining with all above parameters, we obtain from Theorem 3.2 an estimation of the settling time as $T_2 \leq 471$. The result of the finite time median related consensus is shown in Fig. 2. It is apparent that the finite-time consensus is achieved after a transient time $T \approx 1$. The consensus function converges to $g(m(z)) = 0.16$ at time $T' \approx 16$ afterwards. Note that agents move at different speeds in the first phase, while the outlier represents the worst case scenario with respect to the convergence speed. The transient time seems to be dependent of the initial states.

Example 5.2. (Finite-time group consensus over cooperation-competition networks). In this example, we consider multi-agent system (3) with $n = 7$ agents interacting over the communication graphs with two subgraphs shown in Fig. 3. To satisfy the constraint $\rho_k(t) \geq \rho_k \geq 1, \forall t \geq 0, \forall k \in [1, K]$, the system is modeled as a periodic switching grouping, where the underlying communication topology

$$\mathcal{G}(t) = (V, E(t)) = \begin{cases} \{\mathcal{G}_1(t) = (V_1, E_1(t)), \mathcal{G}_2(t) = (V_2, E_2(t))\}, & \text{if } t \in [2k, 2k+1], k \in N, \\ \{\mathcal{G}_3(t) = (V_1, E_3(t)), \mathcal{G}_4(t) = (V_2, E_4(t))\}, & \text{if } t \in [2k-1, 2k], k \in N. \end{cases}$$

Here, the lower bounds of the edge connectivities of the underlying graphs of $\mathcal{G}_1(t)$ and $\mathcal{G}_3(t)$, $\mathcal{G}_2(t)$ and $\mathcal{G}_4(t)$ can be set to $\rho_1 = 1$, $\rho_2 = 2$, respectively. The associated weighted adjacency matrices of $\mathcal{G}_1(t)$, $\mathcal{G}_2(t)$, $\mathcal{G}_3(t)$ and $\mathcal{G}_4(t)$ are all chosen as $w_{ij}(t) = 7i(2 + \sin(t)), \forall (i, j) \in E_1(t), E_2(t), E_3(t), E_4(t)$. The edge-weights between different

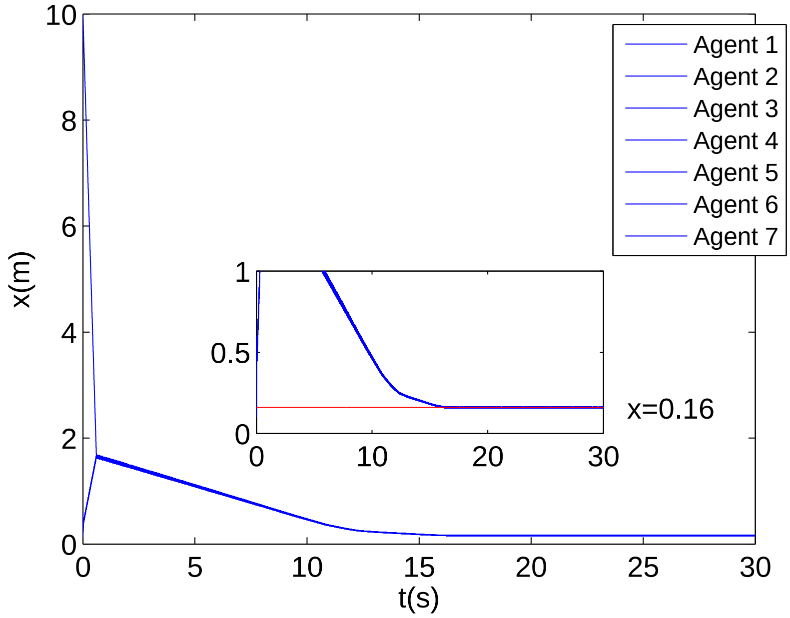


Figure 2. Finite-time median related consensus for multi-agent systems (3), (4) and communication topology shown in Example 1. The initial state is set to $z = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 10]^T$.

subgraphs are taken as

$$w_{ij}(t) = \begin{cases} \frac{1}{3500} i\cos(t)(2+\sin(t)), & \text{if } |i-j| = 3, t \in [2k, 2k+1], k \in N \ , \\ \frac{1}{3500} i\cos(t)(2+\sin(t)), & \text{if } |i-j| = 4, t \in [2k, 2k+1], k \in N \ , \\ 0, & \text{otherwise.} \end{cases}$$

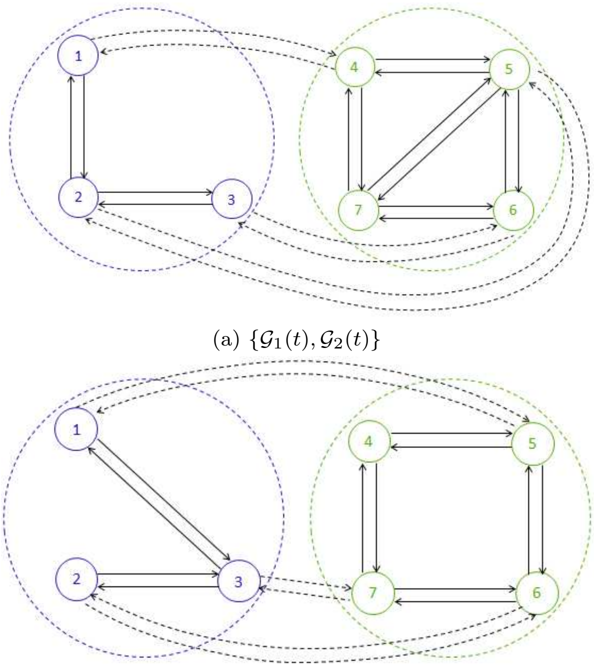
And the edge (i, j) between different subgraphs satisfies that

$$\begin{cases} (i,j) \in E(t), & \text{if } w_{ji}(t) \neq 0 \ , \\ (i,j) \notin E(t), & \text{if } w_{ji}(t) = 0 \ . \end{cases}$$

The constraint $a_i(t)w_{ij}(t) = a_j(t)w_{ji}(t), \forall i \in V, \forall (i,j) \in \cup_{k=1}^{K} E_k(t), \forall t \geq 0$ holds with $a_i(t) = \frac{7}{i(2+\sin(t))}, \forall i \in V, \forall t \geq 0$. By direct calculations, we choose $L_1 = \frac{1}{3}, L_2 = 7, L_3 = 7, L_4 = 0.006, \alpha_{\max} = 1.2, \alpha_{\min} = 1.0$ and $\alpha = [1.018, 1.127, 1.153, 1.096, 1.120, 1.174, 1.195]^T$ which meet all constraints of Theorem 4.3. We consider the case in which one agent holds an outlier initial value in each subgraph. The initial network state is chosen with values in the range $[0.5, 0.9]$ which the initial values of the outlier agents are equal to 0.017 and 10, respectively. The initial network state is thus

$$z = [0.017, 0.5, 0.6, 0.7, 0.8, 0.9, 10]^T,$$

where $z^1 = [0.017, 0.5, 0.6]^T$, $z^2 = [0.7, 0.8, 0.9, 10]^T$. The initial state's average are 0.37 and 3.1, respectively whereas the median value are 0.5 and [0.8,0.9], respectively. Here, $g(\cdot)$ is also chosen as $g(m(z)) = (m(z))^2$. Combining with all above parameters, we



(a) $\{\mathcal{G}_1(t), \mathcal{G}_2(t)\}$

(b) $\{\mathcal{G}_3(t), \mathcal{G}_4(t)\}$

Figure 3. Communication topology for Example 2. The system is modeled as a periodic switching grouping, where it can switch back and forth between $\{\mathcal{G}_1(t), \mathcal{G}_2(t)\}$ and $\{\mathcal{G}_3(t), \mathcal{G}_4(t)\}$. The information exchange among nodes are chosen as $w_{ij}(t) = 7i(2 + \sin(t))$ in the same subgraphs and $\frac{1}{3500} i\cos(t)(2 + \sin(t))$ between different subgraphs.

obtain from Theorem 4.3 an estimation of the settling time as $T_{\max} \leq 108$. The result of the finite time median related group consensus is shown in Fig. 4. As one would expect, the finite-time median related group consensus is realized and the convergence time is estimated as $T_{\max} \approx 6$. Note that the first subgraph with three agents converges to a constant $g(m(z^1)) = 0.25$, and the other subgraph converges to a closed interval $g(m(z^2)) = [0.64, 0.81]$, which conforms to Remark 1.

6. Conclusion

In this paper, we first propose a novel finite-time consensus protocol which achieves agreement with respect to the median related value over collaboration networks. Then we generalize the results to cooperation-competition networks, where the communication network can be divided into several collaboration sub-networks. Some conditions have been derived to choose appropriate gains so that every sub-network reaches an individual median related consensus in finite time. Finally, some numerical simulations are provided to illustrate the obtained theoretical results. For future work, it would be interesting to consider more general directed networks, possibly generalizing the directed graphs with detail-balanced condition. A possible way is to address it by bridging a certain relationship between general directed graphs and the directed graphs with detail-balanced condition. For example, one operation is to set edge-weights of the nonexistent edges to an extremely low number. Median related consensus for multi-agent systems with time-delay and noisy is also a challenging problem to be investigated.

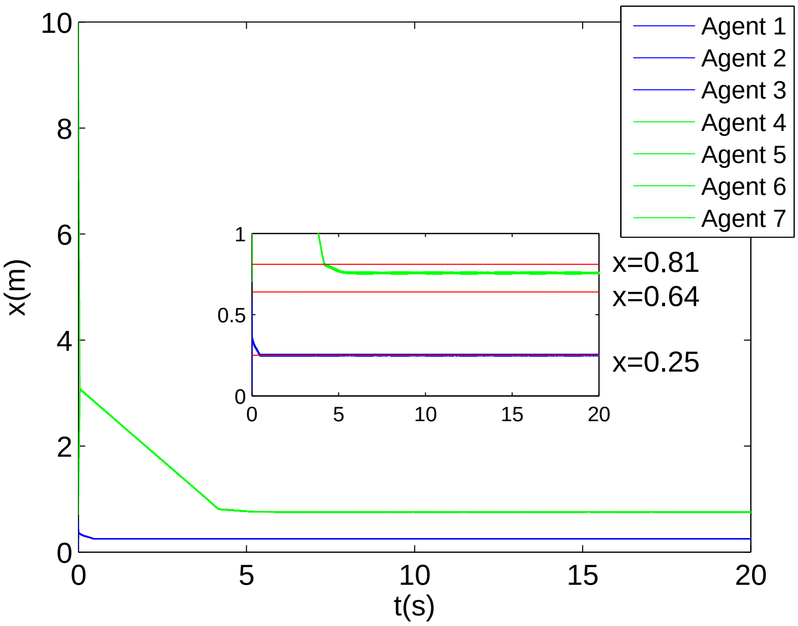


Figure 4. Finite-time median related group consensus for multi-agent systems (3), (4) and communication topology shown in Example 2. The initial state is set to $z = [0.017, 0.5, 0.6, 0.7, 0.8, 0.9, 10]^T$.

Disclosure Statement

The authors declare that there is no conflict of interests regarding the publication of this paper.

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